Cauchy Sequence

Definition:

A sequence (x_n) of $\mathbb{R}$ is said to be a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N$ s.t.

$$|x_n - x_m| < \varepsilon, \ \forall n, m \in N.$$

Theorem:

A sequence of $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

Remark: The Cauchy Convergence Criterion is also equivalent to completeness assumption of $\mathbb{R}$.

Example 1: $\{\frac{1}{n}\}$ is Cauchy.

Example 2: $\{\sqrt{n}\}$ is not Cauchy but $\lim_n(\sqrt{n+1}-\sqrt{n})=0$.

Exercise 1:

Prove the following sequence is convergent:

$$\left\{\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots+\frac{(-1)^{n+1}}{n!}\right\}.$$

Idea: Define $y_n := \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}$. Note that for $n>m$,

$$|y_n-y_m| \le \frac{1}{(m+1)!}+\cdots+\frac{1}{n!} \le \frac{1}{2^m}+\frac{1}{2^{m+1}}+\cdots\frac{1}{2^{n-1}}$$

$$\le \frac{1}{2^m}\left(1+\frac{1}{2}+\cdots\right) \le \frac{1}{2^{m-1}}.$$

Exercise 2:

Prove the harmonic series $(1+\frac{1}{2}+\frac{1}{3}+\cdots)$ diverges.

Idea: Let $x_n := \sum_{k=1}^{n} \frac{1}{k}$.

Note if $n>m$, then

$$|x_n - x_m| = \frac{1}{m+1} + \cdots + \frac{1}{n} \geq \frac{n-m}{n} = 1 - \frac{m}{n}.$$

Can take $\varepsilon = \frac{1}{2} > 0$ and $n = 2m$.

Exercise 3:

If $0<r<1$, & $|x_{n+1} - x_n| < r^n \ \forall n \in \mathbb{N}$.

Show that $\{x_n\}$ is Cauchy.

Idea: If $n>m$,

$$|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$$
$$\leq r^{n-1} + r^{n-2} + \cdots + r^m \leq \frac{r^m}{1-r}.$$

Exercise 4:

Suppose $x_1 < x_2$ & $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n > 2$.

Prove that (x_n) is convergent.

Idea: Note $x_{n+1} - x_n = \frac{x_2 - x_1}{2^{n-1}} \cdot (-1)^{n+1}$. (by induction).

Then $\forall\, n > m \geq N$,

$$
\begin{aligned}
|x_n - x_m| &\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\
&\leq \frac{x_2 - x_1}{2^{n-1}} + \cdots + \frac{x_2 - x_1}{2^{m-1}} < \frac{x_2 - x_1}{2^{m-1}}\left(1 + \frac{1}{2} + \cdots\right) \\
&\leq \frac{x_2 - x_1}{2^{N-2}}.
\end{aligned}
$$

Question:

How to find the limit of (x_n)?